

THE ASYMPTOTIC GROWTH OF THE CONSTANTS IN THE BOHNENBLUST-HILLE INEQUALITY IS OPTIMAL

D. DINIZ, G. A. MUÑOZ-FERNÁNDEZ *, D. PELLEGRINO**, AND J. B. SEOANE-SEPÚLVEDA *

ABSTRACT. We provide (for both the real and complex settings) a family of constants, $(C_m)_{m \in \mathbb{N}}$, enjoying the Bohnenblust–Hille inequality and such that $\lim_{m \rightarrow \infty} \frac{C_m}{C_{m-1}} = 1$, i.e., their asymptotic growth is the best possible. As a consequence, we also show that the optimal constants, $(K_m)_{m \in \mathbb{N}}$, in the Bohnenblust–Hille inequality have the best possible asymptotic behavior. Besides its intrinsic mathematical interest and potential applications to different areas, the importance of this result also lies in the fact that all previous estimates and related results for the last 80 years (such as, for instance, the multilinear version of the famous Grothendieck Theorem for absolutely summing operators) always present constants C_m ’s growing at an exponential rate of certain power of m .

1. PRELIMINARIES. THE HISTORY OF THE PROBLEM

The Bohnenblust–Hille inequality (1931, [2]) asserts that for every positive integer $m \geq 2$ there exists a sequence of non decreasing positive scalars $(C_m)_{m=1}^\infty$ such that

$$(1.1) \quad \left(\sum_{i_1, \dots, i_m=1}^N |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \sup_{z_1, \dots, z_m \in \mathbb{D}^N} |U(z_1, \dots, z_m)|$$

for all m -linear mapping $U : \mathbb{C}^N \times \dots \times \mathbb{C}^N \rightarrow \mathbb{C}$ and every positive integer N , where $(e_i)_{i=1}^N$ denotes the canonical basis of \mathbb{C}^N and \mathbb{D}^N represents the open unit polydisk in \mathbb{C}^N . The original constants obtained by Bohnenblust and Hille are

$$C_m = m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}.$$

Making $m = 2$ in (1.1) we recover Littlewood’s 4/3 inequality (1930, [12]). In the last 80 years, very few improvements for the constants C_m have been achieved:

- $C_m = 2^{\frac{m-1}{2}}$ (Kaijser [10] and Davie [4]),
- $C_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ (Queffélec [16] and Defant & Sevilla-Peris [7]).

Besides their intrinsic mathematical interest, the Bohnenblust–Hille inequality and Littlewood’s 4/3 theorem are extremely useful tools in many areas of Mathematics, just to cite some: Operator theory in Banach spaces, Fourier and harmonic analysis, analytic number theory, etc. (we refer the interested reader to the monographs [1, 11, 15]).

2010 *Mathematics Subject Classification.* 46G25, 47L22, 47H60.

Key words and phrases. Absolutely summing operators, Bohnenblust–Hille Theorem.

*Supported by the Spanish Ministry of Science and Innovation, grant MTM2009-07848.

**Supported by CNPq and PROCAD Novas Fronteiras CAPES.

More recently a new proof of the Bohnenblust–Hille inequality was presented in [6] and this approach, although very abstract, allowed the calculation of new and sharper constants (see [14]). However all the possible constants derived from this approach were calculated by means of recursive formulae, and the expression of these constants as closed formulae seems to be, in most cases, an impossible task. The polynomial version of Bohnenblust–Hille inequality, due to its different nature, presents worse constants and only in 2011 the long standing problem on the hypercontractivity of the constants for the polynomial Bohnenblust–Hille inequality was settled in [5], i.e., the authors proved that (for the polynomial case) there exists $C > 1$ so that

$$\frac{C_m}{C_{m-1}} = C$$

for all $m \geq 1$.

Notwithstanding its eight decades of existence of the Bohnenblust–Hille inequality, the optimal asymptotic behavior of the constants involved is still unknown. From the previous estimates we have

- $C_m/C_{m-1} = \sqrt{2} \approx 1.4142$ (Kaisjer [10] and Davie [4]),
- $C_m/C_{m-1} = \frac{2}{\sqrt{\pi}} \approx 1.1284$ (Queffélec [16] and Defant & Sevilla-Peris [7]).

A very recent numerical study ([13]), presented the possibility of improving the above results (now including the case of real scalars as well) to

$$\lim_{m \rightarrow \infty} \frac{C_m}{C_{m-1}} = 2^{1/8} \approx 1.1090,$$

but this estimate could not be formally proved.

In this article we prove an optimal result for the asymptotic behavior of the Bohnenblust–Hille constants: we provide a family of constants, $(C_m)_{m \in \mathbb{N}}$, satisfying the Bohnenblust–Hille inequality with the best possible asymptotic growth, i.e.,

$$\lim_{m \rightarrow \infty} \frac{C_m}{C_{m-1}} = 1.$$

As a consequence we conclude that the optimal constants in the Bohnenblust–Hille inequality have an optimal behavior, asymptotically speaking.

As we mentioned before, for real scalars the Bohnenblust–Hille inequality is also true; for a long time the best known constants in this setting seemed to be $C_m = 2^{\frac{m-1}{2}}$. For $m = 2$ we have $C_2 = \sqrt{2}$, which is known to be optimal. Thus, it is possible that this absence of improvements for the case of real scalars is motivated by a feeling that the family $C_m = 2^{\frac{m-1}{2}}$ was a good candidate for being optimal. However, as we will see, these constants are quite far from the optimality.

From now on, the letters X_1, \dots, X_m shall stand for Banach spaces and B_{X^*} represents the closed unit ball of the topological dual of X . By $\mathcal{L}(X_1, \dots, X_m; \mathbb{K})$ we denote the Banach space of all continuous m -linear mappings from $X_1 \times \dots \times X_m$ to \mathbb{K} endowed with the sup norm.

If $1 \leq p < \infty$, a continuous m -linear mapping $U \in \mathcal{L}(X_1, \dots, X_m; \mathbb{K})$ is called multiple $(p, 1)$ -summing (denoted by $U \in \Pi_{(p;1)}(X_1, \dots, X_m; \mathbb{K})$) if there exists a constant $K_m \geq 0$ such that

$$(1.2) \quad \left(\sum_{j_1, \dots, j_m=1}^N \left| U(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)}) \right|^p \right)^{\frac{1}{p}} \leq K_m \prod_{k=1}^m \sup_{\varphi_k \in B_{X_k^*}} \sum_{j=1}^N \left| \varphi_k(x_j^{(k)}) \right|$$

for every $N \in \mathbb{N}$ and any $x_{j_k}^{(k)} \in X_k$, $j_k = 1, \dots, N$, $k = 1, \dots, m$. The infimum of the constants satisfying (1.2) is denoted by $\|U\|_{\pi(p;1)}$.

An important simple reformulation of the Bohnenblust–Hille inequality shows that it is equivalent to the assertion that every continuous m -linear form $T : X_1 \times \dots \times X_m \rightarrow \mathbb{K}$ is multiple $(\frac{2m}{m+1}; 1)$ -summing, and the constants involved are the same as those from the Bohnenblust–Hille inequality. Thus, in this article we shall be dealing with the Bohnenblust–Hille inequality in the framework of multiple summing operators.

From now on (and throughout this paper), we let

$$A_p := \sqrt{2} \left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p},$$

where Γ denotes the classical Gamma Function. These constants will appear soon in Theorems 1.1 and 1.2; they are related to the Khinchine inequality and appear in [9], where the optimal constants of the Khinchine inequality are obtained. More details on the nature of A_p will be given in the next section.

The following results (inspired in [6] and in estimates from [9]) were recently proved in [14] by the third and fourth authors:

Theorem 1.1. *For every positive integer m and real Banach spaces X_1, \dots, X_m ,*

$$\Pi_{(\frac{2m}{m+1};1)}(X_1, \dots, X_m; \mathbb{R}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{R}) \text{ and } \|\cdot\|_{\pi(\frac{2m}{m+1};1)} \leq C_{\mathbb{R},m} \|\cdot\|$$

with

$$C_{\mathbb{R},2} = \sqrt{2},$$

$$C_{\mathbb{R},3} = 2^{\frac{5}{6}},$$

$$C_{\mathbb{R},m} = \frac{C_{\mathbb{R},m/2}}{A_{\frac{2m}{m+2}}^{m/2}}$$

for m even and

$$C_{\mathbb{R},m} = \left(\frac{C_{\mathbb{R},\frac{m-1}{2}}}{A_{\frac{2m-2}{m+1}}^{\frac{m-1}{2}}} \right)^{\frac{m-1}{2m}} \cdot \left(\frac{C_{\mathbb{R},\frac{m+1}{2}}}{A_{\frac{2m+2}{m+3}}^{\frac{m+1}{2}}} \right)^{\frac{m+1}{2m}}$$

for m odd.

Theorem 1.2. *For every positive integer m and every complex Banach spaces X_1, \dots, X_m ,*

$$\Pi_{(\frac{2m}{m+1};1)}(X_1, \dots, X_m; \mathbb{C}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{C}) \text{ and } \|\cdot\|_{\pi(\frac{2m}{m+1};1)} \leq C_{\mathbb{C},m} \|\cdot\|$$

with

$$C_{\mathbb{C},m} = \left(\frac{2}{\sqrt{\pi}} \right)^{m-1}$$

for $m \in \{2, 3, 4, 5, 6\}$,

$$C_{\mathbb{C},m} = \frac{C_{\mathbb{C},m/2}}{A_{\frac{2m}{m+2}}^{m/2}}$$

for $m > 6$ even and

$$C_{\mathbb{C},m} = \left(\frac{C_{\mathbb{C}, \frac{m-1}{2}}}{A_{\frac{m+1}{2}, \frac{2m-2}{m+1}}} \right)^{\frac{m-1}{2m}} \cdot \left(\frac{C_{\mathbb{C}, \frac{m+1}{2}}}{A_{\frac{m-1}{2}, \frac{2m+2}{m+3}}} \right)^{\frac{m+1}{2m}}$$

for $m > 5$ odd.

In this article we shall prove that the asymptotic behavior of the above constants is the best possible. This result (the optimality of the asymptotic growth of the constants of Bohnenblust–Hille inequality) seems quite surprising since all the constants involved in all similar results in the theory of multiple summing operators grow at an exponential rate. In fact, this is the case of the previous estimates of the constants of Bohnenblust–Hille inequality from [4, 7, 10, 16] and also the case of the multilinear generalization (for multiple summing operators) of Grothendieck’s theorem for absolutely summing operators. More precisely, the multilinear Grothendieck Theorem [3] states that for every positive integer m , every continuous m -linear operator $U : \ell_1 \times \cdots \times \ell_1 \rightarrow \mathbb{C}$ is multiple $(1; 1)$ -summing and

$$\|U\|_{\pi(1;1)} \leq \left(\frac{2}{\sqrt{\pi}} \right)^m \|U\|.$$

Similar results, which can also be found in [3] (called *coincidence* results), always present constants that, as we mentioned earlier, grow exponentially following the pattern of certain power of m ; this is one of the reasons why we consider the optimality of the asymptotic growth of the constants in the Bohnenblust–Hille inequality quite a surprising result.

2. BOHNENBLUST AND HILLE MEET EULER

In this section we shall see that the limits of the terms that appear in the previous sections are related to Euler’s famous constant γ . Let us recall that γ is classically defined as

$$\gamma = \lim_{m \rightarrow \infty} \left(\sum_{k=1}^m \frac{1}{k} - \log m \right) \approx 0.5772,$$

and it is still not known (since its first appearance in 1734 in Euler’s *De Progressionibus Harmonicis Observationes*) whether this value is algebraic or transcendental.

The results and calculations given in this section are part of the essential tools that shall be used to prove the main result of this article. First of all, and by using the classical known properties of the Gamma Function, we obtain

$$\lim_{x \rightarrow 0} \left(\frac{\Gamma(\frac{1}{2} - x)}{\Gamma(\frac{1}{2})} \right)^{1/x} = 4e^{\gamma},$$

and, thus,

$$\lim_{x \rightarrow 0} \left(\frac{\Gamma(\frac{3}{2} - x)}{\Gamma(\frac{3}{2})} \right)^{1/x} = 4e^{\gamma-2}$$

which gives

$$\lim_{m \rightarrow \infty} \left(\frac{\Gamma(\frac{3m+2}{2m+4})}{\Gamma(\frac{3}{2})} \right)^m = 16e^{2\gamma-4},$$

obtaining

$$\lim_{m \rightarrow \infty} \left[2^{m/4} \cdot \left(\frac{\Gamma\left(\frac{2m}{m+2} + 1\right)}{\sqrt{\pi}} \right)^{(m+2)/4} \right] = \frac{\sqrt{2}}{e^{1-\frac{1}{2}\gamma}}.$$

From [9] it is known that for $1 < p_0 < 2$ so that

$$\Gamma\left(\frac{p_0 + 1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

we have

$$A_p = \sqrt{2} \left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p},$$

whenever $p_0 \leq p < 2$ (numerical calculations estimate $p_0 \approx 1.8474$). So, since

$$\lim_{m \rightarrow \infty} \frac{2m}{m+2} = 2 > p_0,$$

we finally get that (for both, the real and complex settings)

$$(2.1) \quad \lim_{m \rightarrow \infty} \left(A_{\frac{2m}{m+2}}^{m/2} \right)^{-1} = \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \approx 1.4402.$$

Analogously, it can be checked that (for m odd)

$$(2.2) \quad \lim_{m \rightarrow \infty} \left(A_{\frac{2m-2}{m+1}}^{(m+1)/2} \right)^{(-1) \cdot \frac{m-1}{2m}} = \lim_{m \rightarrow \infty} \left(A_{\frac{2m+2}{m+3}}^{(m-1)/2} \right)^{(-1) \cdot \frac{m+1}{2m}} = \frac{e^{\frac{1}{2}-\frac{1}{4}\gamma}}{2^{1/4}} \approx 1.2001.$$

3. THE PROOF

From (2.1) and (2.2) we have (for both, the real and complex settings) that

$$\lim_{n \rightarrow \infty} \frac{C_{2n}}{C_n} = \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}$$

and

$$\lim_{n \rightarrow \infty} \frac{C_{2n+1}}{(C_n)^{\frac{2n}{2(2n+1)}} \cdot (C_{n+1})^{\frac{2n+2}{2(2n+1)}}} = \left(\frac{e^{\frac{1}{2}-\frac{1}{4}\gamma}}{2^{1/4}} \right) \cdot \left(\frac{e^{\frac{1}{2}-\frac{1}{4}\gamma}}{2^{1/4}} \right) = \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}.$$

In other words, for *large* values of n we have the following equivalences

$$\begin{aligned} C_{2n} &\sim \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \right) C_n, \\ C_{2n+1} &\sim \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \right) (C_n)^{\frac{n}{2n+1}} (C_{n+1})^{\frac{n+1}{2n+1}}, \end{aligned}$$

as well from Theorem 1.1, Theorem 1.2, (2.1) and (2.2) we have the next asymptotical identities:

$$(3.1) \quad \frac{C_{2n}}{C_{2n-1}} \sim \frac{C_n}{(C_{n-1})^{\frac{n-1}{2n-1}} (C_n)^{\frac{n}{2n-1}}} \sim \left(\frac{C_n}{C_{n-1}} \right)^{\frac{n-1}{2n-1}}$$

and

$$(3.2) \quad \frac{C_{2n+1}}{C_{2n}} \sim \frac{(C_n)^{\frac{n}{2n+1}} (C_{n+1})^{\frac{n+1}{2n+1}}}{C_n} \sim \left(\frac{C_{n+1}}{C_n} \right)^{\frac{n+1}{2n+1}}.$$

Since $\left(\frac{C_{2n}}{C_n} \right)_{n=1}^{\infty}$ converges, it is bounded by some constant $C > 1$. Let, for $n \in \mathbb{N}$,

$$D_n := \frac{C_{n+1}}{C_n}.$$

Notice also that, since $(C_n)_{n=1}^{\infty}$ is non-decreasing, $D_n \geq 1$. Thus

$$(3.3) \quad C > \frac{C_{2n}}{C_n} = \prod_{j=n}^{2n-1} D_j \geq D_n \geq 1$$

for every n .

Now we show that $(D_n)_{n=1}^{\infty}$ is convergent and that its limit must be 1. We split its proof into four *claims*. The first one of these claims is an almost trivial statement, and we spare the details of its proof to the interested reader:

Claim 3.1. *The sequence $(D_n)_{n=1}^{\infty}$ satisfies the following asymptotical equalities:*

- (1) $D_{2n-1} \sim \sqrt{D_{n-1}}$,
- (2) $D_{2n} \sim \sqrt{D_n}$.

The second of the claims is a crucial tool in order to prove that $(D_n)_{n=1}^{\infty}$ converges to 1:

Claim 3.2. *Let $K > 1$. Suppose that there exists m_0 such that $n > m_0$ implies $1 \leq D_n < K$. Then there exists $m_1 \geq m_0$ such that $n > m_1$ implies $1 \leq D_n < K^{5/8}$.*

Proof. From (1) in Claim 3.1 we know that $\lim_{n \rightarrow \infty} \frac{D_{2n-1}}{\sqrt{D_{n-1}}} = 1$. Given $\varepsilon > 0$ satisfying $\varepsilon < K^{\frac{1}{8}} - 1$ there exists $n_0 > m_0 + 1$ such that $n > n_0$ implies

$$\frac{D_{2n-1}}{\sqrt{D_{n-1}}} < (1 + \varepsilon) < K^{1/8}.$$

Therefore

$$(3.4) \quad D_{2n-1} < K^{1/8} \sqrt{D_{n-1}} < K^{5/8}.$$

In a similar fashion, by (2) in Claim 3.1, there exists $n_1 > m_0 + 1$ such that $n > n_1$ implies

$$\frac{D_{2n}}{\sqrt{D_n}} < (1 + \varepsilon) < K^{1/8}$$

and hence

$$(3.5) \quad D_{2n} < K^{1/8} \sqrt{D_n} < K^{5/8}.$$

Let $n_2 = \max\{n_0, n_1\}$ and $m_1 = 2n_2 + 1$. It follows from (3.4) and (3.5) that $n > m_1$ implies

$$(3.6) \quad 1 \leq D_n < K^{5/8}.$$

To see this let $n > m_1$ and let q be a positive integer such that $n = 2q$ if n is even and $n = 2q - 1$ if n is odd. Now if n is odd we have $2q - 1 = n > m_1 = 2n_2 + 1$, and therefore $q > n_2 > n_0$ and it follows from (3.4) that

$$D_n = D_{2q-1} < K^{5/8}.$$

And if n is even we have $2q = n > m_1 = 2n_2 + 1$, and it follows from (3.5) that

$$D_n = D_{2q} < K^{5/8}.$$

□

Claim 3.3. *Let $L > 1$. Suppose that there exists m_0 such that $n > m_0$ implies $1 \leq D_n < L$. Then there exists $m_2 \geq m_0$ such that $n > m_2$ implies*

$$(3.7) \quad 1 \leq D_n < \sqrt{L}.$$

Proof. From Claim 3.2, with $K = L$, we know that there exists a positive integer m_1 so that $1 \leq D_n < L^{5/8}$ for all $n > m_1$. Using again Claim 3.2, this time with $K = L^{5/8}$, we can assure the existence of a positive integer m_2 such that

$$1 \leq D_n < \left(L^{5/8}\right)^{5/8}$$

for all $n > m_2$. Since $\left(\frac{5}{8}\right)^2 < \frac{1}{2}$ we have (3.7). □

A recursive application of the above result together with the upper bound in (3.3) allows us to obtain a better upper bound (which approaches 1) as n goes to infinity:

Claim 3.4. *Let C be an upper bound for $(D_n)_{n=1}^\infty$ in equation (3.3). For every non negative integer s there exists a positive integer n_0 (depending on s) such that $n > n_0$ implies*

$$1 \leq D_n < C^{2^{-s}}.$$

Proof. We argue by induction on s . The case $s = 0$ is equation (3.3). Suppose that the result holds for some $s \geq 0$. Then there exists n_0 such that $n > n_0$ implies $1 \leq D_n < C^{2^{-s}}$ and, by Claim 3.3, there exists $m_1 > n_0$ such that

$$1 \leq D_n < \left(C^{2^{-s}}\right)^{1/2} = C^{2^{-(s+1)}},$$

whenever $n > m_1$. □

After the four previous claims, we can now easily conclude that $(D_n)_{n=1}^\infty$ is convergent and $\lim_{n \rightarrow \infty} D_n = 1$. In fact, let $\varepsilon > 0$; let s_0 be a positive integer such that $C^{(2^{-s_0})} < 1 + \varepsilon$ and the result follows from Claim 3.4.

REFERENCES

- [1] R. Blei, *Analysis in integer and fractional dimensions*, Cambridge Studies in Advanced Mathematics, vol. 71, Cambridge University Press, Cambridge, 2001.
- [2] H. F. Bohnenblust and E. Hille, *On the absolute convergence of Dirichlet series*, Ann. of Math. (2) **32** (1931), no. 3, 600–622.
- [3] F. Bombal, D. Pérez-García, and I. Villanueva, *Multilinear extensions of Grothendieck’s theorem*, Q. J. Math. **55** (2004), no. 4, 441–450.
- [4] A. M. Davie, *Quotient algebras of uniform algebras*, J. London Math. Soc. (2) **7** (1973), 31–40.
- [5] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaies, and K. Seip, *The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive*, Ann. of Math. (2) **174** (2011), 485–497.
- [6] A. Defant, D. Popa, and U. Schwaerting, *Coordinatewise multiple summing operators in Banach spaces*, J. Funct. Anal. **259** (2010), no. 1, 220–242.
- [7] A. Defant and P. Sevilla-Peris, *A new multilinear insight on Littlewood’s 4/3-inequality*, J. Funct. Anal. **256** (2009), no. 5, 1642–1664.

- [8] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.
- [9] U. Haagerup, *The best constants in the Khintchine inequality*, Studia Math. **70** (1981), no. 3, 231–283 (1982).
- [10] S. Kaijser, *Some results in the metric theory of tensor products*, Studia Math. **63** (1978), no. 2, 157–170.
- [11] H. König, *Eigenvalue distribution of compact operators*, Operator Theory: Advances and Applications, vol. 16, Birkhäuser Verlag, Basel, 1986.
- [12] J. E. Littlewood, *On bounded bilinear forms in an infinite number of variables*, Q. J. Math. **1** (1930), 164–174.
- [13] G. A. Muñoz-Fernández, D. Pellegrino, and J. B. Seoane-Sepúlveda, *Estimates for the asymptotic behavior of the constants in the Bohnenblust–Hille inequality*, Linear Multilinear A., In Press.
- [14] D. Pellegrino and J. B. Seoane-Sepúlveda, *New upper bounds for the constants in the Bohnenblust–Hille inequality*, J. Math. Anal. Appl., In Press.
- [15] A. Pietsch, *Eigenvalues and s -numbers*, Cambridge Studies in Advanced Mathematics, vol. 13, Cambridge University Press, Cambridge, 1987.
- [16] H. Queffélec, *H. Bohr's vision of ordinary Dirichlet series: old and new results*, J. Anal. **3** (1995), 43–60.

UNIDADE ACADEMICA DE MATEMÁTICA E ESTATÍSTICA,
 UNIVERSIDADE FEDERAL DE CAMPINA GRANDE,
 CAIXA POSTAL 10044,
 CAMPINA GRANDE, 58429-970, BRAZIL.
E-mail address: `diogo@dme.ufcg.edu.br`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,
 FACULTAD DE CIENCIAS MATEMÁTICAS,
 PLAZA DE CIENCIAS 3,
 UNIVERSIDAD COMPLUTENSE DE MADRID,
 MADRID, 28040, SPAIN.
E-mail address: `gustavo_fernandez@mat.ucm.es`

DEPARTAMENTO DE MATEMÁTICA,
 UNIVERSIDADE FEDERAL DA PARAÍBA,
 58.051-900 - JOÃO PESSOA, BRAZIL.
E-mail address: `pellegrino@pq.cnpq.br`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,
 FACULTAD DE CIENCIAS MATEMÁTICAS,
 PLAZA DE CIENCIAS 3,
 UNIVERSIDAD COMPLUTENSE DE MADRID,
 MADRID, 28040, SPAIN.
E-mail address: `jseoane@mat.ucm.es`